MAL 111: INTRODUCTION TO ANALYSIS & DIFFERENTIAL EQUATIONS

MAJOR

MAX. MARKS. 50

Questions 1-4 carry 5 marks and Questions 5-9 carry 6 marks each.

(1) A point $x \in \mathbb{R}$ is said to be a dyadic rational if $x = \frac{p}{2^q}$ for some integers p and q. Let $0 < p < 2^q$, then x is a dyadic rational fraction. Show that the dyadic rational fractions are dense in $[0,1]$, i.e. given any $a < b \in [0,1]$ there exists a dyadic rational fraction x such that $a < x < b$.

[Hint: x is a dyadic rational fraction if and only if there exists a natural number $\mathcal{N}$ such that its binary expansion $0.r_1r_2\ldots r_k\ldots$ has $r_n = 0$ for $n > \mathcal{N}$]

(2) Let $f, g \in \mathcal{R}[a,b]$ be such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there exists an $x \in [a,b]$ for which $f(x) = g(x)$.

(3) Let f be defined on $(-1,1)$ such that $f^{(n)}$ exists and is continuous on $(-1,1)$. If $f(0) = 2,\ f'(0) = -3,\ f^{(2)}(0) = 1,\ f^{(3)}(0) = 6$, then for each $x \in (-1,1)$, $x \neq 0$, show that there is a y between 0 and x such that $f^{(4)}(y)$ is a function of x. Determine this function.

(4) Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain(open, connected), with $n \geq 2$ and let $f : \mathcal{D} \to \mathbb{R}$ be continuous. If $f(\mathbf{a}) \neq 0$ at an interior point $\mathbf{a} \in \mathcal{D}$, then prove that there exists an r-ball $B(\mathbf{a}, \mathbf{r})$ in $\mathcal{D}$ throughout which f has the same sign as $f(\mathbf{a})$.

(5) (a) Let A and B be connected subsets of the metric space (X, d). If $A \cap B \neq \emptyset$, then show that $A \cup B$ is connected.
(b) Will $A \cap B$ be always connected for connected A and B given that $A \cap B \neq \emptyset$? Discuss.

(6) Which of the following subsets of $\mathbb{R}$ are compact:

- $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$
- $[0,1] \cup [3,4]$
- $\cup_{r=1}^{\infty} [2r, 2r+1]$
- rationals in $[0,1]$
- $\{0,1,2,3,4,5,6,7,8,9\}$
- $\mathbb{R}$

Write a line or two to justify your assertion in each case.

Date: 22nd NOVEMBER, 2011

(7) Let

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or if } y \geq x^2; \\ 1 & \text{if } 0 < y < x^2. \end{cases}$$

Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any straight line through the origin. Find a curve through the origin along which f has the constant value 1 except at $(0,0)$. Is f continuous at $(0,0)$? Discuss.

(8) Prove that if $f(x,y)$ satisfies Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ and if $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, then the function $\phi(x,y) = f(u(x,y), v(x,y))$ is also a solution to the Laplace's equation.

(9) (a) (i) Discuss the maximum - minimum values of the function $f(x,y) = xy$.

(ii) Find the general solution of the differential equation $y' + (\cos x)y = \sin x \cos x$.

OR

(b) Construct the first few Picard iterates of the initial value problem

$$y' = 2xy, \quad y(0) = 1$$

and show that they converge to the solution of the given initial value problem.